Module 4

The Time Value of Money

FINE 3010, Financial Management

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Finance Is Time Travel

- “The essence of finance is time travel. Saving is about moving resources from the present into the future; financing is about moving resources from the future back into the present...”
- Matt Levine on Bloomberg

Goals

- **Part 1**
  - Translate dollars today into dollars tomorrow ("future value")
  - Translate dollars tomorrow into dollars today ("present value")
  - Financial calculators

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- **Part 2**
  - Special cases, perpetuities, annuities, and their variations
  - The role of inflation
  - EAR vs. APR

Module 4, Part 1

Future Value, Present Value, and Financial Calculators

The Time Value of Money

Would you rather receive a dollar today, or a dollar tomorrow?

The Time Value of Money

- A dollar today is worth more than a dollar tomorrow

The Time Value of Money

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- Why?

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- Assuming,
  - There is some positive rate of return investment for us to pursue
  - No behavioral concerns that make us poor with money

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Today (time passes) Later

. . .

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- Compound interest is the default in finance (e.g., mortgage rate, deposit rate)

Future Value, Compound Interest

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- How much interest will you have earned after five years?

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Year	Start balance	Compound interest	End balance
1	\$100.00	$0.06 \times 100.00 = \$6.00$	\$106.00
2	\$106.00	$0.06 \times 106.00 = \$6.36$	\$112.36
3	\$112.36	$0.06 \times 112.36 = \$6.74$	\$119.10
4	\$119.10	$0.06 \times 119.10 = \$7.15$	\$126.25
5	\$126.25	$0.06 \times 126.25 = \$7.57$	\$133.82

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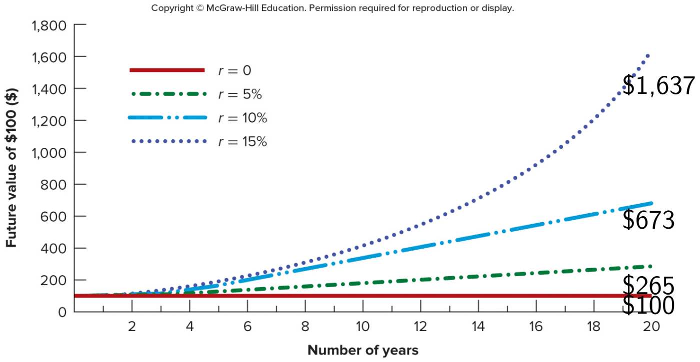
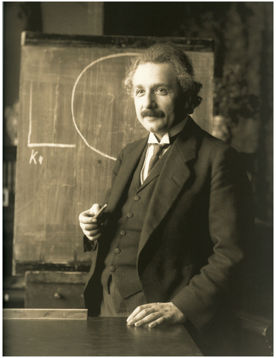
- With compound interest, your \$100 grows to $\$100(1+r)^T = 133.82$, so you earned \$33.82 in compound interest

Future Value

- The future value (FV) depends heavily on the interest rate r, especially as the time dimension t grows



What's the Most Powerful Force in the Universe?



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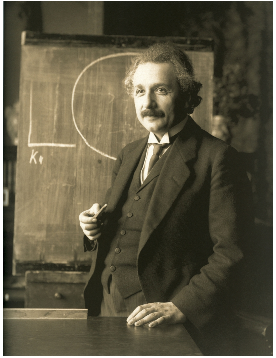

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- These financial returns are not directly related in any way with **Module 3**'s accounting-based ROA and ROE measures

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- Confirm, you can also verify that $(1 + 0.132)^2 - 1 = 28.2\%$

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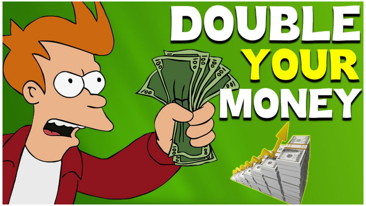
- a. $\text{FV} = \$24 \times (1 + 0.08)^{395} = \$382$ trillion (roughly the same as total world wealth)
- b. $\text{FV} = \$24 \times (1 + 0.035)^{395} = \$19$ million

Example, Future Value

- Suppose you earn an annual return of 6%. How many years will it take until your money doubles?



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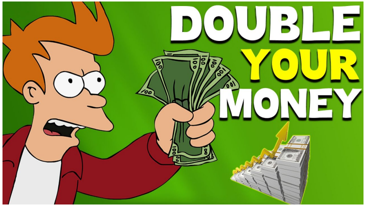
- Suppose you earn an annual return of 6%. How many years will it take until your money doubles?
- 11.896 years, because $(1 + 0.06)^{11.896} = 2$



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- **Discount rate**, the rate used to compute PVs (i.e., r)
  - For a guaranteed future cash flow, the correct discount rate is the risk-free interest rate
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- **Discounted cash flow (DCF)**, the method of calculating present values by discounting future cash flows

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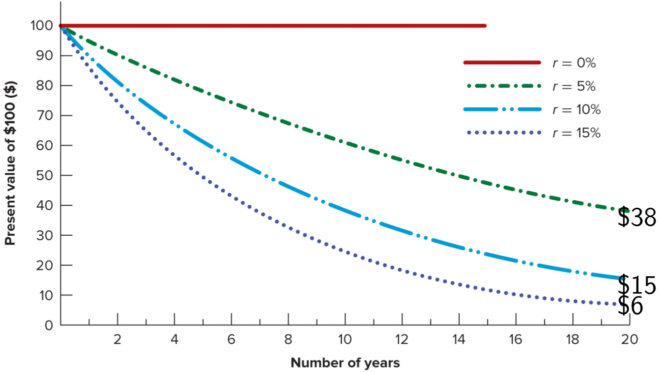

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$$\text{PV}_{\text{total}} = \frac{C_1}{(1+r)^1} + \frac{C_2}{(1+r)^2} + \cdots$$

- where C_1 is the cash flow in one year, C_2 is the cash flow in two years, etc.

Example, Multiple Future Cash Flows

- Your auto dealer gives you the choice to either
  - A. Pay \$15,500 cash now, or
  - B. Make three payments, \$8,000 now, \$4,000 one year from now, and \$4,000 two years from now
- If the discount rate is 8%, which do you prefer?

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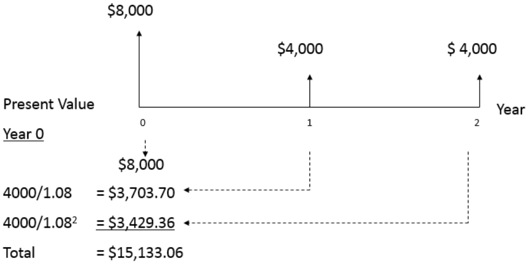

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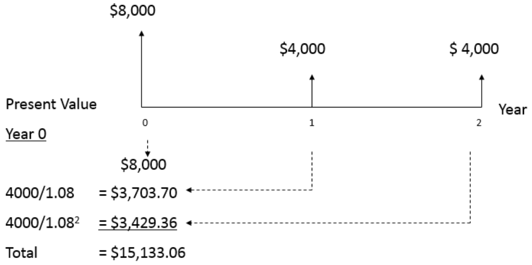

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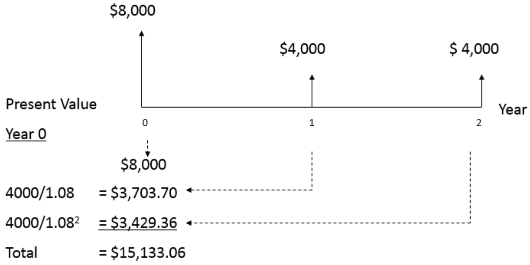

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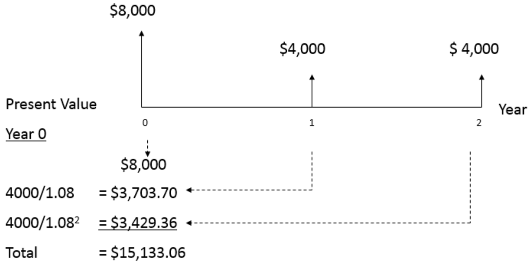

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- The present value ($15,133.06) is less than $15,500, so choose the payment plan
- This is the principle behind the NPV rule in capital budgeting, which we will return to in **Modules 6 and 7**

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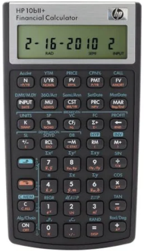
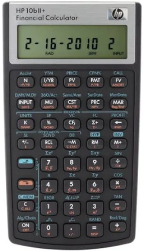
Financial Calculators

- Speeds up time value of money problems (and solves harder ones!)



Financial Calculators

- Speeds up time value of money problems (and solves harder ones!)
- A complement to the underlying math and intuition, never a substitute



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- After you reset, you should repeat the next two settings!

Set the Number of Significant Digits to at Least 4!

- On the HP 10bII+, to change the number of digits displayed to 4 decimals, press
  - OrangeOption
  - = (DISP)
  - 4

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- On the HP 10bII+, to set it to 1, press
  - 1
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- To change back to the factory default (12), press
  - 12
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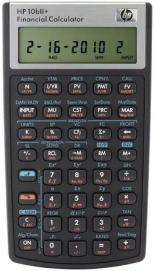
- Note, monthly compounding assumes the interest rate is provided as an APR (we will talk more about APRs soon)

Financial Calculators, Basic Functionality

- N, number of periods (usually years)



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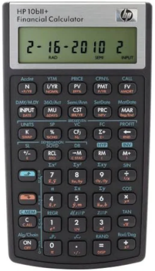

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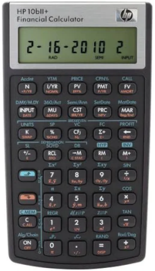

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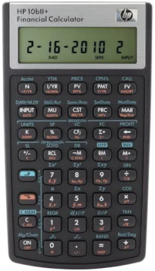

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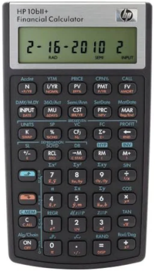

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  - 3000 FV
  - PV

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- We solved this problem earlier in this module using the following formula

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- So the PV will have the opposite sign of the payments (PMT) and the future value (FV)

Example, Puerto Rico Borrows

- In 2007, Puerto Rico needed to borrow \$2.6 billion for 47 years
- It did so by selling bonds that promised to pay \$1,000 after 47 years (no periodic "coupon" payments)
- These bonds were sold at \$94.40 each (they sold around \$2.6 billion$/94.40 = 27.5$ million bonds, and thus promised to pay back around \$27.5 billion in total)
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- Bonus, what would the payment be if the monthly interest was 0.2%?

Example, Pick Your Payment!

- Many car dealerships offer "Pick Your Payment", you can choose how much to pay (PMT) every month! What usually varies is N (the number of months payments continue)
- Suppose Joe and Lucy both buy cars for \$20,000, and their monthly interest is 0.6% (just over 7% per year). Joe picks a payment of \$400, Lucy picks a payment of \$600. How many months will Joe and Lucy be making payments on their cars?

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- Let's try to solve some of the previous examples using Excel!

Example, A-Rod's $275 Million Contract with the Yankees

Year	Salary	"Signing bonus"
2008	27	2
2009	32	1
2010	32	1
2011	31	1
2012	29	1
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2014	25	3
2015	21	
2016	20	
2017	20	
Total	**265**	**10**

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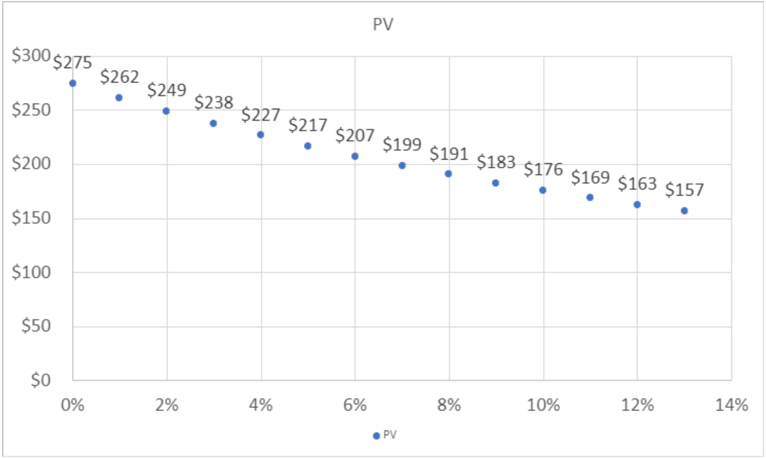
- What was A-Rod's $275 million contract with the Yankees actually worth?
- Suppose the discount rate is 10%. What is the present value of his contract?
- Why might it be useful for A-Rod to know the PV, and not simply the (undiscounted) sum of all future payments?

The PV of Payments to A-Rod



Module 4, Part 2

Perpetuities, Annuities, Inflation, and EAR vs. APR

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- Why care about perpetuities and annuities?
  - Many real-world cash flows have these patterns (particularly annuities, e.g., most mortgages or other fixed-term loans)
  - There exist neat shortcuts that make it very easy to calculate the present value of these patterns of cash flows

Perpetuity Structure

- A perpetuity is a stream of equal cash flows (C) that occur at regular intervals and last forever

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- C is the annual cash flow (aka the payment), and r is the discount rate

Example, Perpetuities

- Recall A-Rod's Yankees contract. Suppose the discount rate is 10%, then the PV of the contract (paying \$275m over 10 years) is \$175.6m
- Now suppose the Yankees offer A-Rod two options,
  - A. The contract as previously described (10 years with \$275m in total), or
  - B. A perpetual payment of C dollars every year. He will play for 10 years, but continue to be paid forever, as long as he lives, and then his children, and then his children's children, etc.
- How much should the Yankees offer as the perpetual payment C for him to be **indifferent** between contract (A) and the "being paid forever" contract (B)?

Example, Perpetuities

- Solution, he is indifferent if the contracts have the same present value!

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$$\text{PV}_{\text{Contract A}} = \text{PV}_{\text{Contract B}} = \frac{C_B}{r} \quad \Rightarrow \quad C_B = \text{PV} \times r = 175.6 \times 10\% = \$17.56m$$

Perpetuity, Timing

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- **Ordinary perpetuity**, first payment at time 1 $\qquad \text{PV} = \frac{C}{r}$

- **Delayed perpetuity**, first payment at time $t > 1$ $\qquad \text{PV} = \frac{C}{r} \times \frac{1}{(1+r)^{t-1}}$

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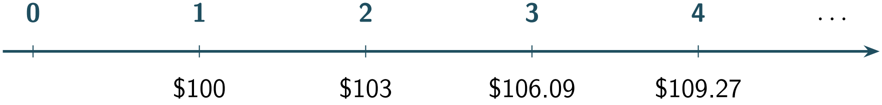

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0	1	2	3	4	...
	\$100	\$103	\$106.09	\$109.27	

$$\text{PV of growing perpetuity} = \frac{\text{Cash flow}}{\text{Discount rate} - \text{Growth rate}} = \frac{C}{r-g}$$

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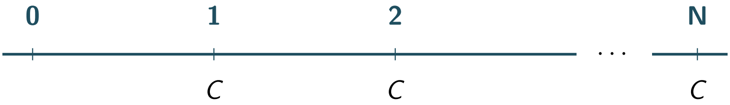
$$\text{PV of growing perpetuity} = \frac{\text{Cash flow}}{\text{Discount rate} - \text{Growth rate}} = \frac{C}{r-g}$$

- This formula comes up often when valuing stocks, e.g., in the "Gordon growth model" and DCF valuation (more in **Modules 5 and 9**)
- When $g = 0$, the PV simplifies to the formula for a regular (non-growing) perpetuity

Annuities

- An annuity is a stream of N equal cash flows paid at regular intervals



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0	**1**	**2**	$\cdots$	**N**
	C	C		C

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$$\text{PV of annuity} = C \underbrace{\frac{1}{r}\left(1 - \frac{1}{(1+r)^N}\right)}_{\text{present value annuity factor (PVAF)}}$$

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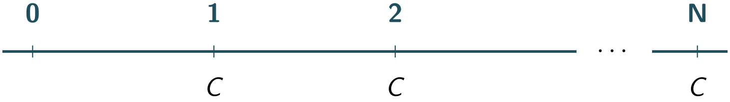

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- C is the annual cash flow (aka the payment), r is the discount rate, and N or T is the number of years that the annuity lasts

Example, Annuity

- You are purchasing a car. You are scheduled to make 3 annual installments of \$8,000 per year, with the first payment a year from now. Given a discount rate of 10%, what is the price you are paying for the car (i.e., what is the PV of the annuity)?

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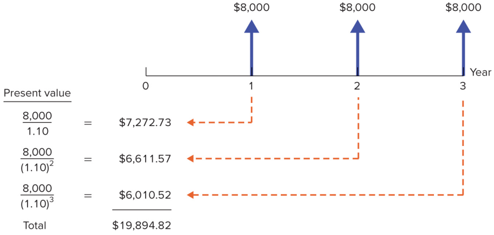

Example, Annuity

- Suppose the discount rate for A-Rod's contract is 10%, so the present value is $175.6m
- Now suppose instead the Yankees offer an annuity for 10 years with the same present value
- How much would he get paid per year, and in total (undiscounted) over 10 years, under such a same-value annuity contract?

Example, Annuity

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$$175.6 = C\,\frac{1}{10\%}\left(1 - \frac{1}{(1+10\%)^{10}}\right) \quad \Rightarrow \quad C = 28.58\text{m}$$

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- So he would be paid \$28.58m per year, or \$285.8m in undiscounted total over 10 years (slightly more in total, because his actual contract is front-loaded, which raises its present value compared to a constant payment)

Discuss, Front-Loaded or Level Cash Flows?

- If we have two cash flows with the same undiscounted total, but one is front-loaded, another is equally distributed over the years (i.e., an annuity), which one has a higher present value?

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- Answer, the front-loaded one
- What about a back-loaded cash flow and an annuity?

Perpetuities and Annuities, Timing

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- **Delayed annuity**, first payment at time $t > 1$ $\quad\quad \text{PV} = \text{PV annuity} \times \frac{1}{(1+r)^{t-1}}$

Practice Problem

- A store offers two payment plans. Under the installment plan, you pay 25% down and 25% of the purchase price in each of the next 3 years. If you pay the entire bill immediately, you can get a discount of 7% on the purchase price. Assume the product sells for \$100

a-1. Calculate the present value of the payments if you can borrow or lend funds at an interest rate of 4%

a-2. Which is a better deal?

b-1. Calculate the present value if the payments on the 4-year installment plan do not start for a full year

b-2. Which is a better deal?

Practice Problem

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- On a financial calculator, $\text{PMT} = -25$, $\text{N} = 3$, $\text{FV} = 0$, $\text{I/YR} = 4$, leading to 69.38, and next, $69.38 + 25 = 94.38$

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Practice Problem

- A local bank advertises the following deal, "Pay us \$100 at the end of each year for 11 years, and then we will pay you (or your beneficiaries) \$100 at the end of each year forever"

a. Calculate the present value of your payments to the bank if the interest rate is 7.00%

b. What is the present value of a \$100 perpetuity deferred for 11 years if the interest rate is 7.00%?

c. Is this a good deal?

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$$PV = 100\left(\frac{1}{7\%}\right)\left(1 - \frac{1}{(1+7\%)^{11}}\right) = \$749.83$$

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$$PV_{11} = \frac{C}{r}$$

$$PV_{11} = \frac{100}{7\%} = \$1{,}428.57$$

$$PV = \frac{PV_{11}}{(1+r)^{11}} = \frac{1{,}428.57}{(1+7\%)^{11}} = \$678.70$$

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- c. Not a good deal (you pay \$749.83 for something worth \$678.70)

Growing Annuities

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$$\text{PV of growing annuity} = \frac{C}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^{N}\right]$$

Take the Lump Sum or Growing Annuity?

- The Robinsons of Munford, Tennessee were one of the three winners in the “1.6 billion” Powerball jackpot in 2016
- They could choose between,
  - A. A lump sum of \$327 million, or
  - B. 30 installments of a growing annuity due with $g = 4\%$, starting at \$9.43m, with 30 payments, totaling around \$529 million

Photo, AP/Mark Humphrey

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- Which option should they take?

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Take the Lump Sum or Growing Annuity?

- Not evenly divided, but a growing annuity due ($g = 4\%$), it starts immediately and grows for another 29 years at 4% and then ends

Year	Payout
0	9.43
1	9.81
2	10.20
...	...
27	27.19
28	28.28
29	29.41
Total	**528.88**

Take the Lump Sum or Growing Annuity?

- Not evenly divided, but a growing annuity due ($g = 4\%$), it starts immediately and grows for another 29 years at 4% and then ends
- Suppose the discount rate is 3%

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...	...
27	27.19
28	28.28
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- The PV of the growing annuity due is
$$\frac{C}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^N\right]\times(1+r)$$

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$$=\frac{9.43}{0.03-0.04}\left[1-\left(\frac{1+0.04}{1+0.03}\right)^{30}\right]\times(1+0.03)$$

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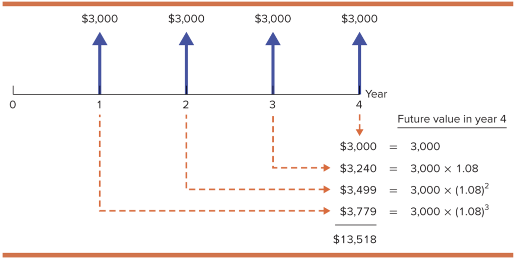
- That is, the present value of the annuity is almost exactly the same as (but slightly lower than) the $327m lump sum

Future Value of Annuities

FIGURE 5.11 **Calculating the future value of an ordinary annuity of $3,000 a year for four years (interest rate = 8%).**



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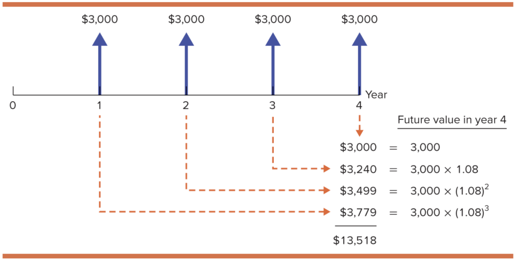


$$\text{FV of annuity of \$1 a year} = \text{PV of annuity of \$1 a year} \times (1+r)^t$$

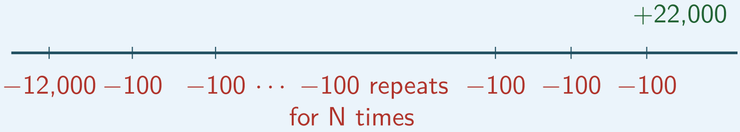
$$= \left[\frac{1}{r} - \frac{1}{r(1+r)^t}\right] \times (1+r)^t = \frac{(1+r)^t - 1}{r}$$

Practice Problem

- I now have \$12,000 in the bank earning interest of 0.50% per month. I need \$22,000 to make a down payment on a house. I can save an additional \$100 per month. How long will it take me to accumulate the \$22,000?

Practice Problem

- From your perspective, the cash flow is



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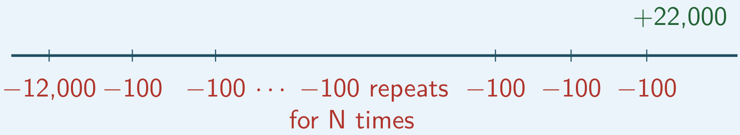

- You need to solve, future value (FV) of the \$12,000 + FV of the N \$100s = FV of the \$22,000

$$12000 \cdot (1+0.005)^N + 100 \cdot \frac{(1+0.005)^N - 1}{0.005} = 22000$$

Practice Problem

- The problem can, alternatively, be solved as, present value (PV) of the \$12,000 + PV of the N \$100s = PV of the \$22,000

Enter		0.50	−12,000	−100	22,000
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Solve for	54.52				

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- Let's recall that financial calculators are programmed to solve the following,

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- You just turned 25. You have found a well-paying job and been inspired by the "FIRE" (financial independence, retire early) movement. You plan to save $100,000 every year for 20 years and then retire when you turn 45
- Given a 10% annual return, how much will you have saved by the time you plan to retire in 20 years?

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- Solution, step 1, calculate the present value of your annuity savings

$$\text{PV} = \$100{,}000 \, \frac{1}{0.10}\left(1 - \frac{1}{(1+0.10)^{20}}\right) = \$100{,}000 \times 8.514 = \$851{,}400$$

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- Step 2, multiply by $(1+r)^t$ to get the future value of your savings

$$\text{FV} = \text{PV} \times (1+r)^{20} = \$851{,}400 \times (1+0.10)^{20} = \$5.727 \text{ million}$$

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- That is, you save \$2,000,000 undiscounted ($100{,}000 \times 20$), worth \$851,400 today, and that money grows by a factor of $(1+0.10)^{20} = 6.7275$ over 20 years, so you will have \$5.727 million to retire on

Example, Annuities and Retirement Savings

- You are currently 25 years old, and you want to retire in 20 years and withdraw \$100,000 per year for 35 years while in retirement. How much do you have to save each year over the next 20 years to achieve this? Assume the return and discount rate is 10%

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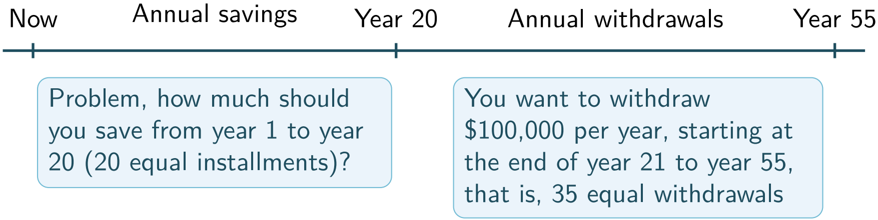

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$$\text{PV}_{\text{withdrawals, today}} = \text{PV}_{\text{withdrawals, year 20}} \times \frac{1}{(1+10\%)^{20}} = \$143{,}351$$

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- Step 3, calculate how much to save over the next 20 years to have that present value

$$\$\text{AnnualSavings} \, \frac{1}{0.10}\left(1 - \frac{1}{(1+0.10)^{20}}\right) = \$143{,}351 \;\Rightarrow\; \text{AnnualSavings} = \$16{,}838$$

Practice Problem

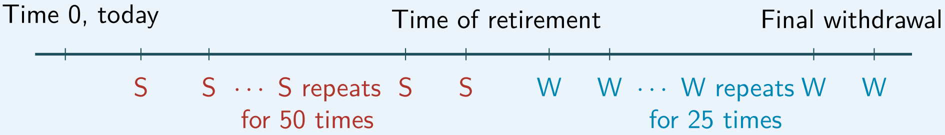
- A couple will retire in 50 years. They plan to spend about \$20,000 a year in retirement, which should last about 25 years. They believe that they can earn a real interest rate of 9% on retirement savings

a. If they make annual payments into a savings plan, how much will they need to save each year? Assume the first payment comes in 1 year

b. How would the answer to part (a) change if the couple also realize that in 20 years they will need to spend \$50,000 on their child's college education?

Practice Problem

- a. The cash flow is



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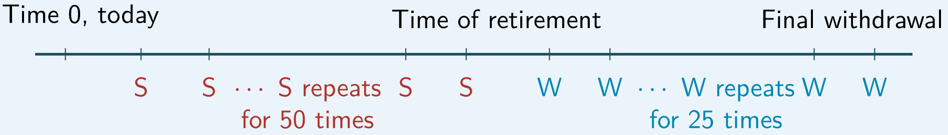

- How much savings is required at the time of retirement?

$$\text{PV}_{\text{withdrawals, year 50}} = 20{,}000\,\frac{1}{0.09}\left(1 - \frac{1}{(1+0.09)^{25}}\right) = 196{,}451.59$$

- On a calculator, $\text{PMT} = 20{,}000$, $\text{I/YR} = 9$, $\text{N} = 25$, $\text{FV} = 0$

Practice Problem

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$$\text{FV}_{\text{savings, year 50}} = S\left(\frac{(1+0.09)^{50}-1}{0.09}\right) = 196{,}451.59 \;\Rightarrow\; S = \$241.02$$

- On a calculator, FV = 196,451.59, I/YR = 9, N = 50, PV = 0

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$$PV = \frac{50{,}000}{1.09^{20}} + \frac{196{,}451.59}{1.09^{50}} = 11{,}563.53$$

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$$PV = \frac{50{,}000}{1.09^{20}} + \frac{196{,}451.59}{1.09^{50}} = 11{,}563.53$$

- Now determine the annual savings for 50 years that equates to that present value

$$\$11{,}563.53 = C \times \frac{1 - \frac{1}{(1+9\%)^{50}}}{9\%} \Rightarrow C = \$1{,}054.90$$

- On a calculator, PV = 11,563.53, I/YR = 9, N = 50, FV = 0

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- Timing,
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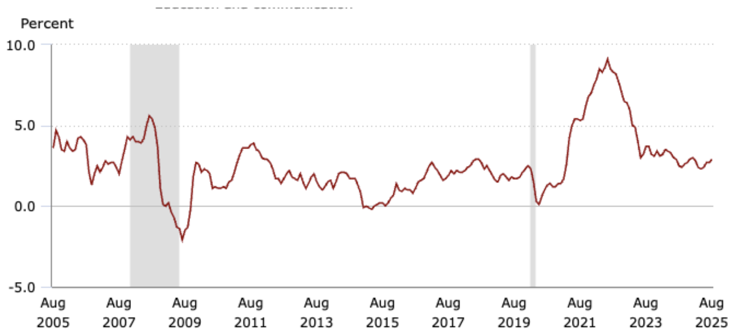
  - Perpetuity and annuity dues make their first payment at $t = 0$
  - Delayed perpetuities and annuities make their first payment at $t > 1$

History of Inflation



Note: Shaded area represents recession, as determined by the National Bureau of Economic Research.
Source: U.S. Bureau of Labor Statistics.

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Inflation and Returns

- In the presence of inflation, an investor's real return (e.g., the real interest rate) is always less than the nominal return

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- Inflation also affects discounting,
  - We should discount nominal dollar cash flows using a nominal discount rate
  - We should discount real dollar cash flows using a real discount rate

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- Similarly,

$$r_{\text{annual}} = (1 + r_{\text{semiannual}})^{2} - 1 \qquad r_{\text{annual}} = (1 + r_{\text{daily}})^{365} - 1$$

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- Add 1 to each side, raise both sides to the power of 1/12, and then subtract 1 again from each side

$$r_{\text{monthly}} = (1 + r_{\text{annual}})^{1/12} - 1 = (1 + 10\%)^{1/12} - 1 = 0.797\%$$

Interest Rates for Most Financial Products Are Cited as "APRs"

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CITI DISCLOSURES

Interest Rates and Interest Charges	
Annual Percentage Rate (APR) for Purchases	**15.24% to 25.24%,** based on your creditworthiness. These APRs will vary with the market based on the Prime Rate.[a]
APR for Balance Transfers	**0%** introductory APR for 18 months from date of first transfer when transfers are completed within 4 months from date of account opening. After that, your APR will be **15.24% to 25.24%**, based on your creditworthiness. These APRs will vary with the market based on the Prime Rate.[a]
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Penalty APR and When it Applies	Up to **29.99%**, based on your creditworthiness. This APR will vary with the market based on the Prime Rate.[c] This APR may be applied to your account if you: (1) Make a late payment or (2) Make a payment that is returned. **How Long Will the Penalty APR Apply?** If your APRs are increased for either of these reasons, the Penalty APR may apply indefinitely.

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- APR ignores compounding. Don't be fooled!
- How much are you actually paying in interest if you borrow on your credit card? Not the APR! We call the "actual" interest the effective annual rate (EAR) when distinguishing it from APR

$$\text{EAR} = (1 + 0.01)^{12} - 1 = 0.1268 = 12.68\% \quad \text{(this is what you're actually paying)}$$

EAR vs. APR, Definitions

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- Note, EAR is how we should calculate interest rates, with compounding! APR can be misleading but is often quoted in practice

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Compounding	Periods (m)	Rate	Growth factor	EAR
Annually	1	6%	1.06	6.0000%
Semiannually	2	3	$1.03^2 = 1.0609$	6.0900
Quarterly	4	1.5	$1.015^4 = 1.061364$	6.1364
Monthly	12	0.5	$1.005^{12} = 1.061678$	6.1678
Weekly	52	0.11538	$1.0011538^{52} = 1.061800$	6.1800
Daily	365	0.01644	$1.0001644^{365} = 1.061831$	6.1831

Example, EAR vs. APR

- Bank 1 pays an APR of 12% compounded semi-annually
- Bank 2 pays an APR of 11.9% compounded daily
- We deposit \$100. In which bank account will the money grow more? What are the EARs of each bank?

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$$\text{Bank 1 earns you } \$100 \times \left(1 + \frac{0.12}{2}\right)^2 = \$112.36$$

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- The EARs are 12.36% and 12.63% respectively. A higher EAR means more interest!

Practice Problem

- Professor's Annuity Corporation offers a lifetime annuity to retiring professors. For a payment of \$79,000 at age 65, the firm will pay the retiring professor \$575 a month until his death

a. If the professor's remaining life expectancy is 20 years, what is the monthly interest rate on this annuity?

b. What is the effective annual interest rate?

c. If the monthly interest rate is 1.00%, what monthly annuity payment can the firm offer to the retiring professor?

Practice Problem

- a. The monthly payments are a 240-month ordinary annuity

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$$\$575 \times \left[\frac{1}{r} - \frac{1}{r \times (1+r)^{240}}\right] = \$79{,}000 \Rightarrow r = 0.52\%$$

- On a calculator, PMT $= 575$, N $= 240$, PV $= -79{,}000$, FV $= 0$. Find I/YR

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- b.

$$\text{EAR} = (1 + 0.0052)^{12} - 1 = 0.0638 = 6.38\%$$

- To compute $(1.0052)^{12}$, you can take either route, (1) PV $= 1$, I/YR $= 0.52$, PMT $= 0$, N $= 12$, find FV, which yields 1.0638, subtract 1, or (2) use the power function, 1.0052, then x^y, 12, $=$, which yields 1.0638, subtract 1

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- c.

$$C \times \left[\frac{1}{0.01} - \frac{1}{0.01\times(1.01)^{240}}\right] = \$79{,}000 \Rightarrow C = PMT = \$869.86$$

- On a calculator, N = 240, PV = −79,000, FV = 0, I/YR = 0.1. Find PMT

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- Perpetuities and annuities have shortcut PV formulas, mind the timing of the first payment
- The real return strips inflation out of the nominal return
- EAR captures compounding while APR does not, so compare rates on an EAR basis

Thank you!

Please let me know if you have any questions.